Smith Normal Form and Combinatorics

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Smith normal form

$\boldsymbol{A}$: $n \times n$ matrix over commutative ring $\boldsymbol{R}$ (with 1)

Suppose there exist $\boldsymbol{P}, \boldsymbol{Q} \in \mathrm{GL}(n, R)$ such that Smith normal form (SNF) of A.

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Note. (1) Can extend to $m \times n$.

$$(2)\ \mathrm{unit} \cdot \det(A) = \det(B) = d_1^n d_2^{n-1} \cdots d_n.$$

Thus SNF is a refinement of det.

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- April 1883: shared *Grand prix des sciences mathématiques* with Minkowski

Row and column operations

From now on, assume $R = \mathbb{Z}$.

Can put a matrix into SNF by the following operations.

- Add a multiple of a row to another row.
- Add a multiple of a column to another column.
- Multiply a row or column by a **unit** in R.

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Over a field, SNF is **row reduced echelon form** (with all unit entries equal to 1).

Existence of SNF

Theorem (**Smith**). A has a unique SNF up to units.

Algebraic interpretation of SNF

A: an $n \times n$ matrix over $\mathbb{Z}$ with rows
$v_1, \dots, v_n \in R^n$

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$\mathbb{Z}^n/(v_1, \dots, v_n)$: **(Kasteleyn) cokernel** of A

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minor: determinant of a square submatrix

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Special case: e_1 is the gcd of all entries of A.

Laplacian matrices

$\boldsymbol{L(G)}$: Laplacian matrix of the graph G

rows and columns indexed by vertices of G

$$\boldsymbol{L}(G)_{uv} = \begin{cases} -\#(\text{edges } uv), & u \neq v \\ \deg(u), & u = v. \end{cases}$$

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reduced Laplacian matrix $\boldsymbol{L_0(G)}$: for some vertex v, remove from $\boldsymbol{L}(G)$ the row and column indexed by v

Matrix-tree theorem

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Applications to sandpile models, chip firing, etc.

An example

Reduced Laplacian matrix of K_4:

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

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What about SNF?

An example (continued)

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ -4 & 4 & -1 \\ 8 & -4 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ -4 & 4 & 0 \\ 8 & -4 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 0 \\ 4 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Reduced Laplacian matrix of K_n

$$\begin{aligned} \mathbf{L_0(K_n)} &= nI_{n-1} - J_{n-1} \\ \det L_0(K_n) &= n^{n-2} \end{aligned}$$

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Theorem. $\boldsymbol{L_0}(K_n) \xrightarrow{\text{SNF}} \mathrm{diag}(1, n, n, \ldots, n)$, *a refinement of Cayley's theorem that* $\kappa(K_n) = n^{n-2}$.

Proof that $L_0(K_n) \xrightarrow{\text{SNF}} \mathrm{diag}(1, n, n, \ldots, n)$

Trick: 2×2 submatrices (up to row and column permutations):

$$\begin{bmatrix} n-1 & -1 \\ -1 & n-1 \end{bmatrix}, \quad \begin{bmatrix} n-1 & -1 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

with determinants $n(n-2)$, $-n$, and 0. Hence $e_1 e_2 = n$. Since $\prod e_i = n^{n-2}$ and $e_i | e_{i+1}$, we get the SNF $\mathrm{diag}(1, n, n, \ldots, n)$.

The n-cube

C_n: graph of the n-cube

The *n*-cube

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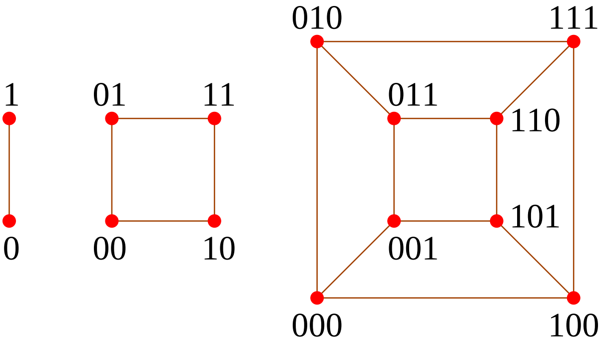

An open problem

$$\kappa(C_n) = 2^{2^n-n-1} \prod_{i=1}^{n} i^{\binom{n}{i}}$$

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2-adic behavior is **unknown**.

SNF of random matrices

with **Yinghui Wang**

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Huge literature on random matrices, mostly connected with eigenvalues.

Not much work on SNF of random matrices over $\mathbb{Z}$.

Is the question interesting?

$\mathbf{Mat}_k(n)$: all $n \times n$ $\mathbb{Z}$-matrices with entries in $[-k, k]$ (uniform distribution)

$p_k(n, d)$: probability that if $M \in \mathrm{Mat}_k(n)$ and $\mathrm{SNF}(M) = (e_1, \ldots, e_n)$, then $e_1 = d$.

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Theorem. $\lim_{k\to\infty} p_k(n, d) = \dfrac{1}{d^{n^2}\zeta(n^2)}$

Sample result

$\boldsymbol{\mu_k(n)}$: probability that the SNF of a random $A \in \mathrm{Mat}_k(n)$ satisfies $e_1 = 2$, $e_2 = 6$.

$$\boldsymbol{\mu(n)} = \lim_{k \to \infty} \mu_k(n).$$

Conclusion

$$
\begin{aligned}
\mu(n) &= 2^{-n^2}\left(1-\sum_{i=(n-1)^2}^{n(n-1)} 2^{-i}+\sum_{i=n(n-1)+1}^{n^2-1} 2^{-i}\right) \\
&\cdot\frac{3}{2}\cdot 3^{-(n-1)^2}(1-3^{(n-1)^2})(1-3^{-n})^2 \\
&\cdot\prod_{p>3}\left(1-\sum_{i=(n-1)^2}^{n(n-1)} p^{-i}+\sum_{i=n(n-1)+1}^{n^2-1} p^{-i}\right).
\end{aligned}
$$

Cyclic cokernel

$\boldsymbol{\kappa(n)}$: probability that an $n \times n$ $\mathbb{Z}$-matrix has SNF $\mathrm{diag}(e_1, e_2, \ldots, e_n)$ with $e_1 = e_2 = \cdots = e_{n-1} = 1$

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Theorem. $$\kappa(n) = \frac{\prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n}\right)}{\zeta(2)\zeta(3)\cdots}$$

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Corollary (**T. Ekedahl**, 1991)

$$\lim_{n\to\infty} \kappa(n) = \frac{1}{\zeta(6)\prod_{j\geq 4}\zeta(j)}$$

$$\approx 0.846936\cdots .$$

Small number of generators

g: number of generators of cokernel (number of entries of SNF $\neq$ 1) as $n \to \infty$

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$$1 - (3.46275\cdots)2^{-(\ell+1)^2}(1 + O(2^{-\ell}))$$

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$$3.46275\cdots = \frac{1}{\prod_{j\geq 1}\left(1-\frac{1}{2^j}\right)}$$

Two Catalan determinants

Theorem. *There is a unique sequence* $a_0, a_1, \ldots$ *of integers satisfying*

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ a_2 & a_3 & \cdots & a_{n+2} \\ \vdots & \vdots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & \cdots & a_{n+1} \\ a_2 & a_3 & \cdots & a_{n+2} \\ a_3 & a_4 & \cdots & a_{n+3} \\ \vdots & \vdots & & \vdots \\ a_{n+1} & a_{n+2} & \cdots & a_{2n+1} \end{vmatrix} = 1.$$

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Namely, $a_n = \frac{1}{n+1}\binom{2n}{n}$ *(***Catalan number***).*

A q-analogue

Let

$$\mathcal{A}_n = \{(i_1, \ldots, i_n) : 0 \le i_1 \le \cdots \le i_n, \quad i_k \le k-1\},$$

so $\#\mathcal{A}_n = C_n$. Define the **q-Catalan polynomial**

$$\tilde{C}_n(q) = \sum_{(i_1,\ldots,i_n)\in\mathcal{A}_n} q^{\binom{n}{2}-(i_1+\cdots+i_n)}.$$

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Example. $\mathcal{A}_3 = \{000, 001, 011, 002, 012\}$, so

$$\tilde{C}_3(q) = q^3 + q^2 + 2q + 1.$$

Ramanujan

$$C_n(q) = q^{\binom{n}{2}} \tilde{C}_n(1/q)$$

$$\begin{aligned} F(q,x) &:= \sum_{n \geq 0} C_n(q) x^n \\ &= \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{qx}{1 - \cfrac{q^2 x}{1 - \cdots}}}} \end{aligned}$$

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$$e^{-2\pi/5} F(e^{-2\pi}, -e^{-2\pi}) = \frac{1}{\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{1+\sqrt{5}}{2}}.$$

Determinant and SNF

$$\boldsymbol{U_n} = [\tilde{C}_{i+j}(q)]_{i,j=0}^{n}, \quad \boldsymbol{V_n} = [\tilde{C}_{i+j+1}(q)]_{i,j=0}^{n}$$

Theorem (**Fürlinger-Hofbauer** 1985, **Cigler** 1999).

$$\begin{aligned} \det U_n &= q^{n(n+1)(4n-1)/6} \\ \det V_n &= q^{n(n+1)(4n+5)/6} \end{aligned}$$

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Theorem (**Bessenrodt-RS** 2015). Over the ring $\mathbb{Z}[q]$ we have

$$\begin{aligned} U_n &\overset{\text{snf}}{\to} \mathrm{diag}(1, q, q^6, q^{15}, \ldots, q^{\binom{2n}{2}}) \\ V_n &\overset{\text{snf}}{\to} \mathrm{diag}(1, q^3, q^{10}, q^{21}, \ldots, q^{\binom{2n+1}{2}}). \end{aligned}$$

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Special case of a much more general multivariate result.

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