Hyperfinite graphs
and combinatorial optimization

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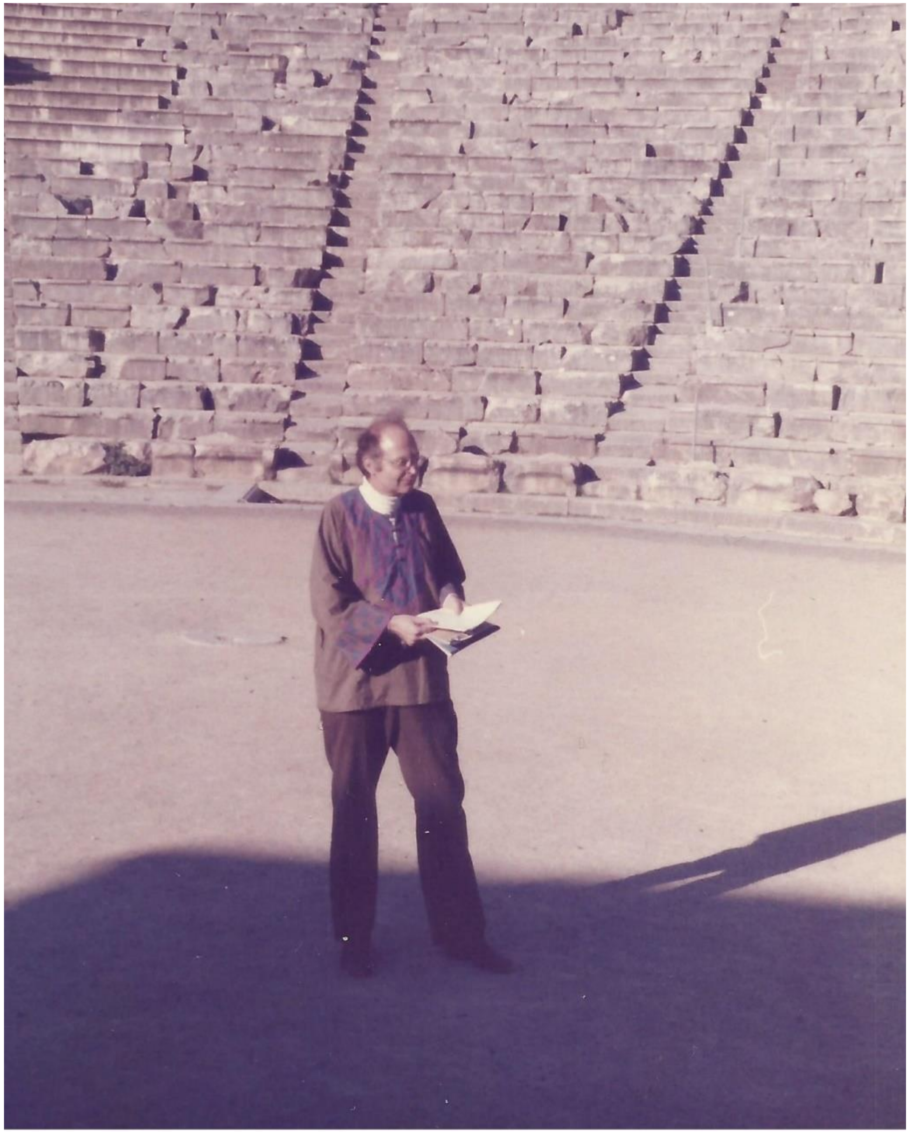
Don in Epidauros 1995



Graphing, definition

Bounded degree ($\leq D$) Borel graph

on standard probability space V (say [0,1])

with "double counting" condition:

$$\int_A \deg(x,B)\,dx = \int_B \deg(x,A)\,dx = \eta(A\times B)$$

Extends to measure η on Borel subsets of V^2.

„edge measure"

Why graphings?

„The notion of an algorithm is basic to

all of computer programming, so we should

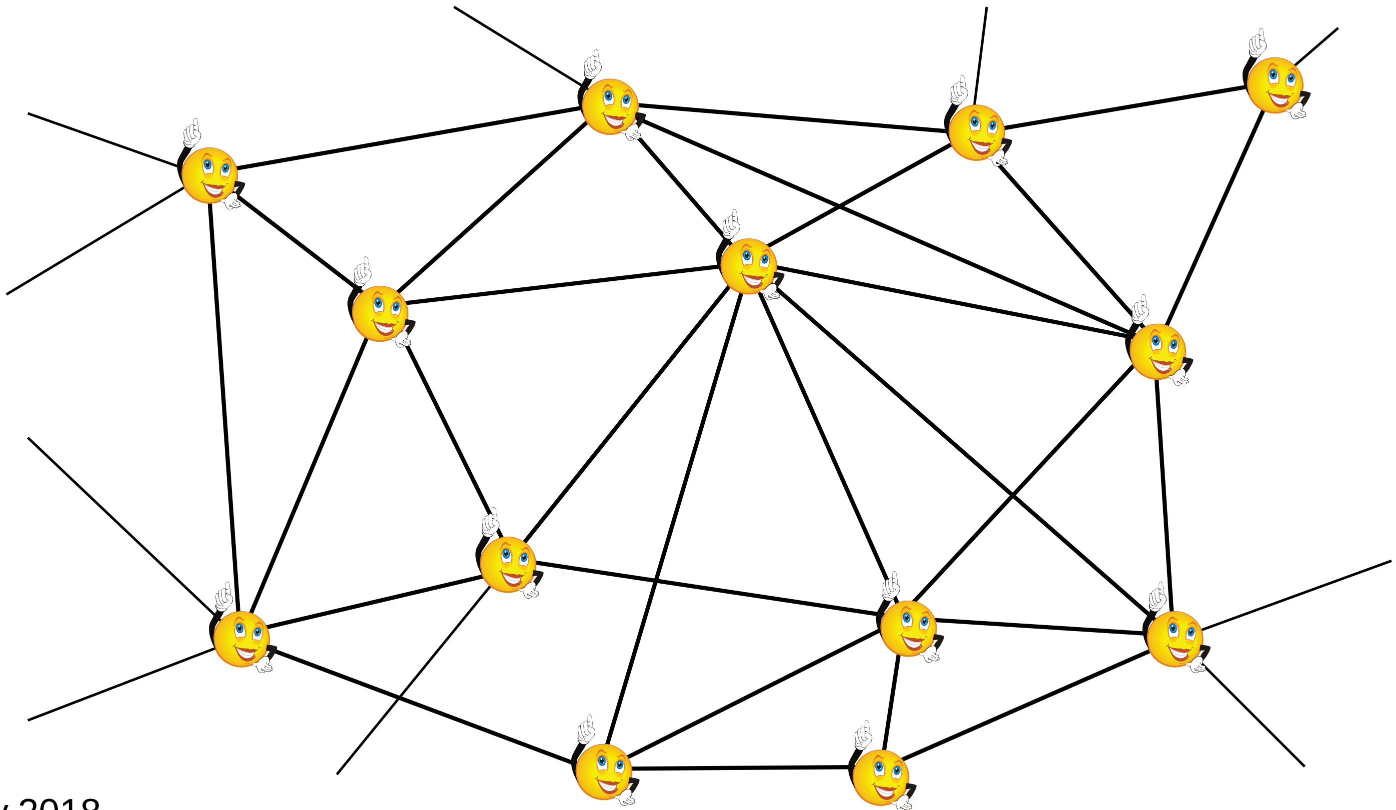
begin with a careful analysis of this concept.”

Donald Knuth

Why graphings?

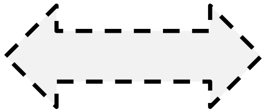
Find a perfect matching in G.

distributed algorithm: Nguyen and Onak

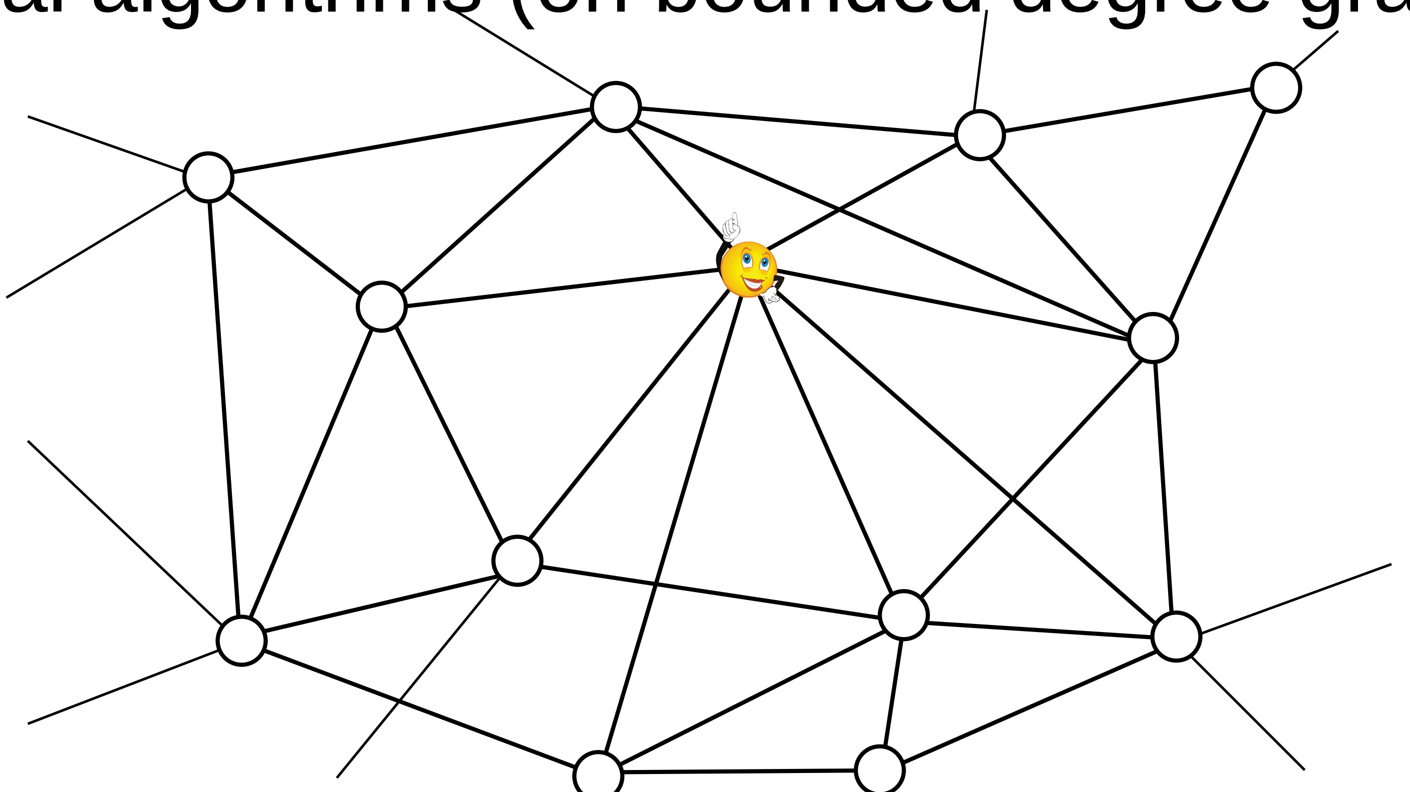


Why graphings?

distributed algorithms

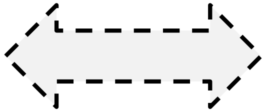


local algorithms (on bounded degree graphs)

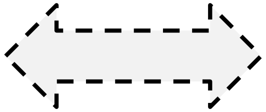


Why graphings?

distributed algorithms



local algorithms (on bounded degree graphs)



property testing (on bounded degree graphs)

Why graphings?

distributed algorithms

local algorithms (on bounded degree graphs)

property testing (on bounded degree graphs)

local algorithms on graphings

The infinite is a good approximation of the large finite.

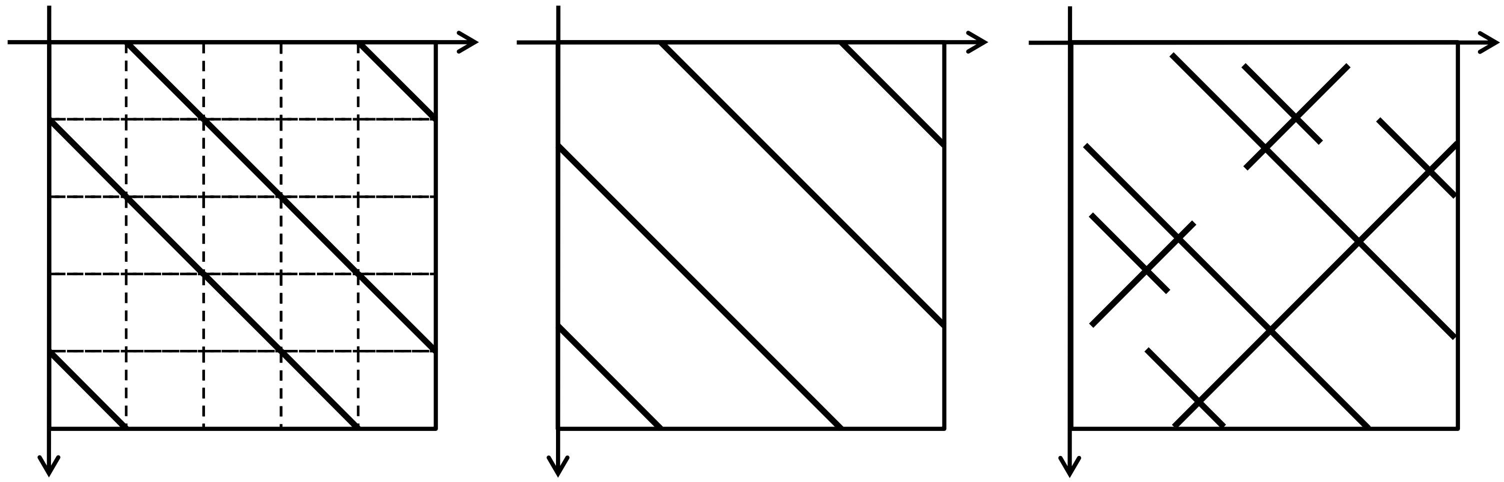
Why graphings?

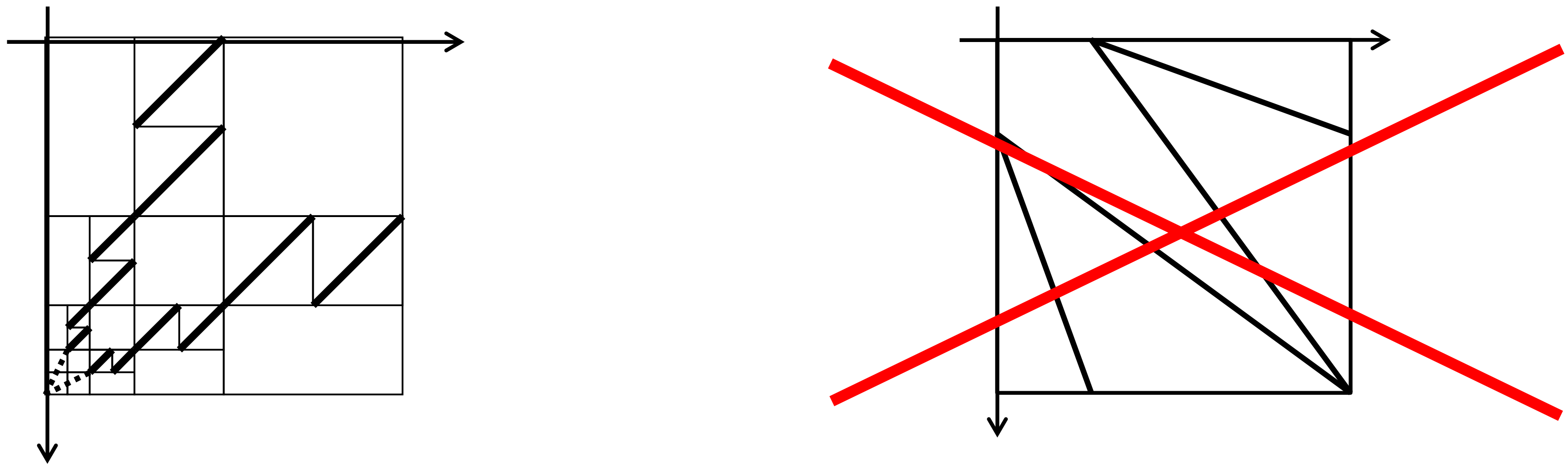
Ergodic theory

Finitely generated groups

Limits of convergent graphs sequences
with bounded degree

Graphing, examples

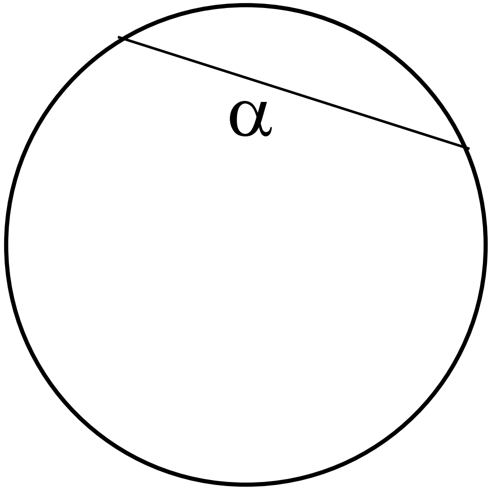




Graphing, examples

unit circumference

α irrational



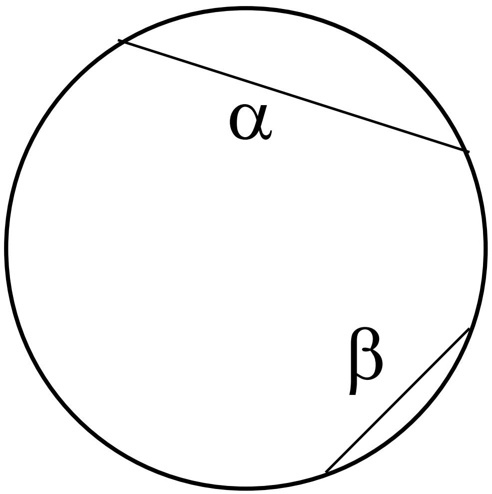
components:

2-way infinite paths

Graphing, examples

unit circumference

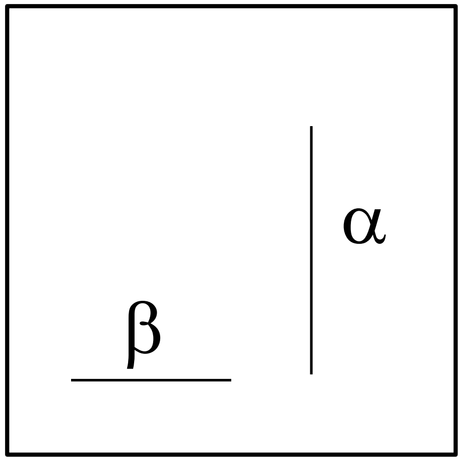
α,β irrational, lin. indep



components: grids

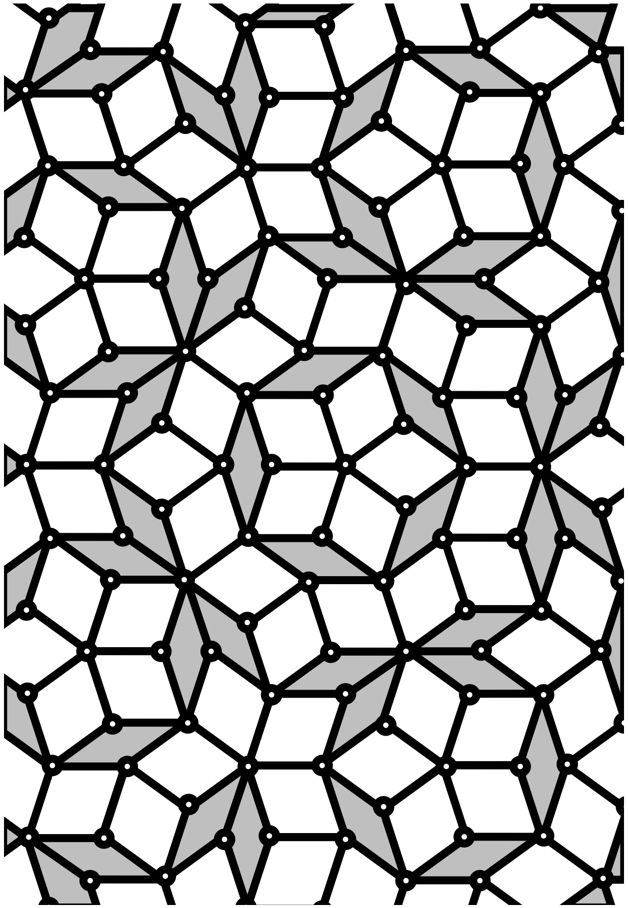
1x1 torus

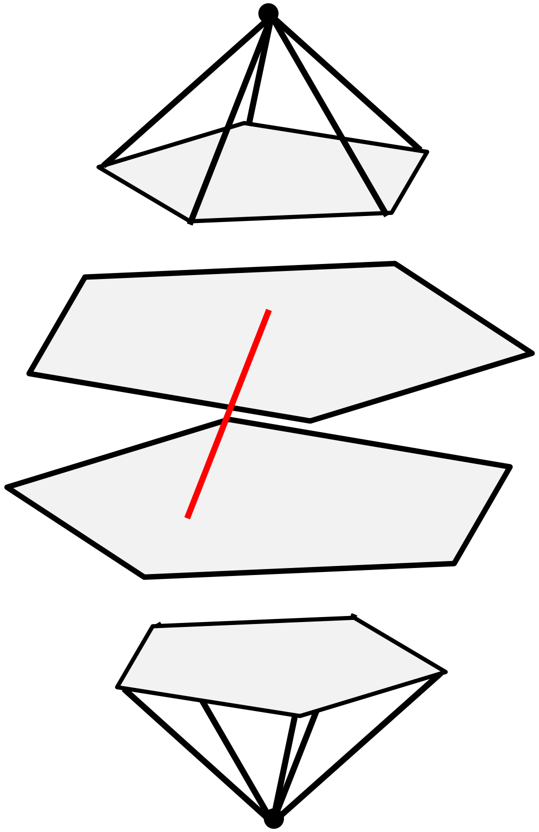
α,β irrational



components: grids

Penrose tilings





rhombic icosahedron

de Bruijn - Bárász

Local equivalence, definition

$x \in V \quad \mapsto \quad G_x$: component of x

uniform random $x \quad \mapsto \quad G_x$: random connected rooted (countable) graph

unimodular random network

Benjamini - Schramm

Local equivalence, definition

$x \in V \quad \mapsto \quad G_x$: component of x

G_1, G_2 locally equivalent:

for random uniform $x \in V$,

distributions of $(G_1)_x$ and $(G_2)_x$

are the same

Local isomorphism, definition

$\varphi : V(G_1) \to V(G_2)$ local isomorphism:

measure preserving and

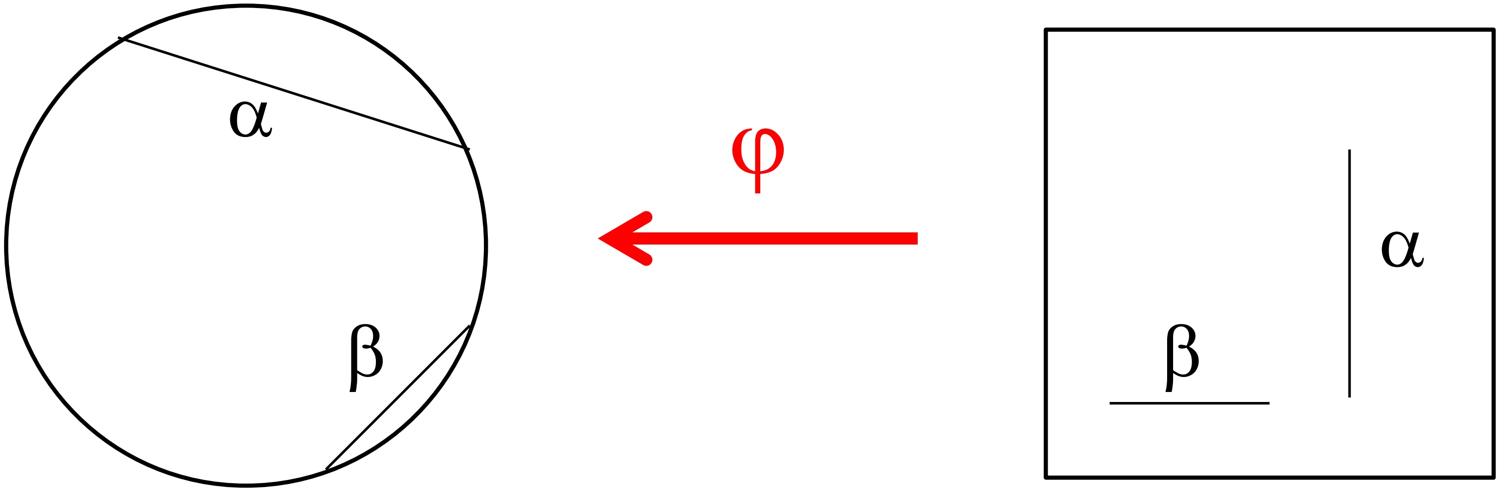
$(\forall x)$ isomorphism between

$(G_1)_x$ and $(G_2)_{\varphi(x)}$

Existence of local isomorphism

proves local equivalence.

Local isomorphism, example



components: grids

components: grids

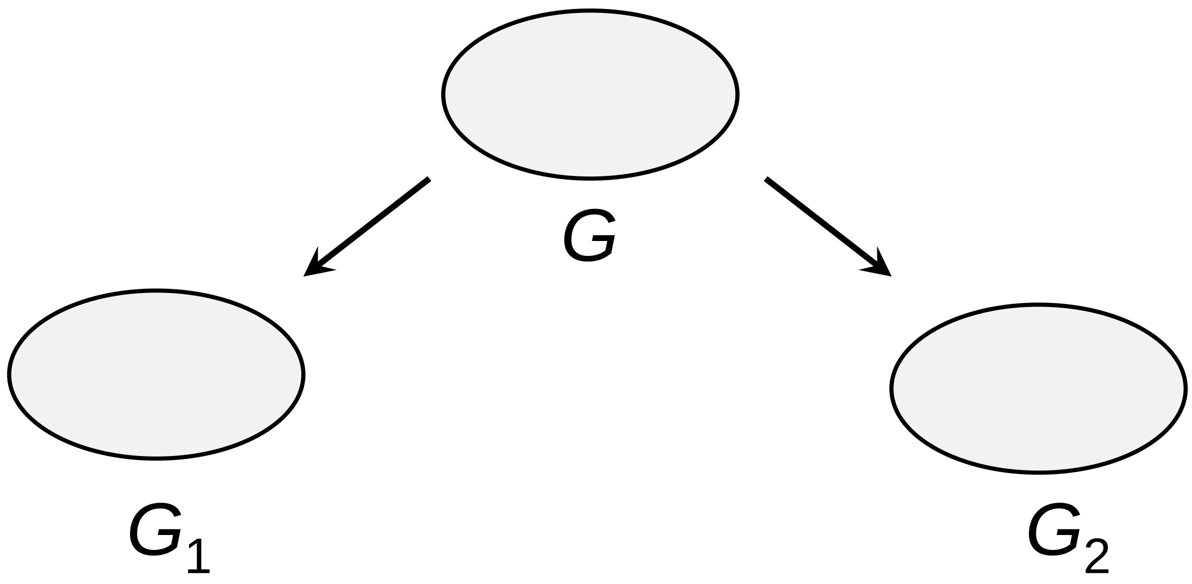
$(x,y) \mapsto x+y \bmod 1$

Local equivalence

G_1 and G_2 are locally equivalent

$\Leftrightarrow$

$\exists G$ and local isomorphisms $G \rightarrow G_1$, $G \rightarrow G_2$.



Graph partition problem

k-edge-separator: $T \subseteq E(G)$, $\forall$component of G-T has $\leq k$ nodes

Finite graph:

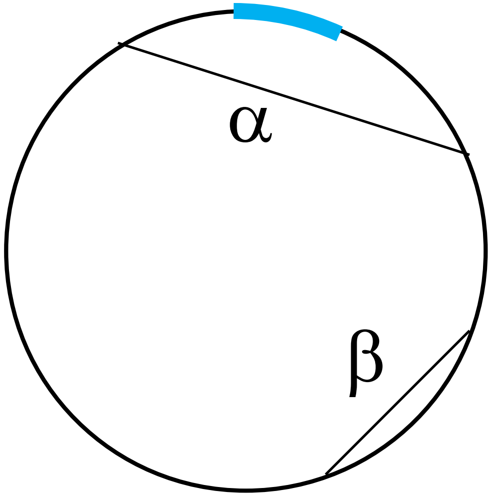
$$\mathrm{sep}_k(G) = \min\left\{\frac{|T|}{n} : T \ k\text{-edge-separator}\right\}$$

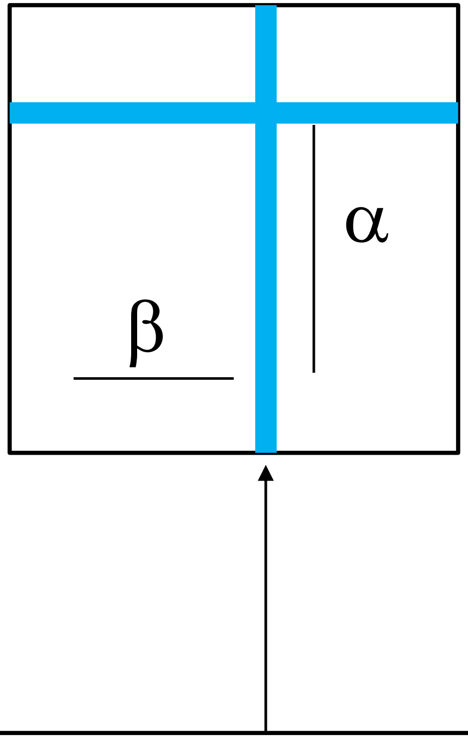
Graphing:

$$\mathrm{sep}_k(G) = \min\{\eta(T) : T \ k\text{-edge-separator}\}$$

Graphing G hyperfinite: $\mathrm{sep}_k(G) \to 0 \quad (k \to \infty)$

Hyperfinite graphings, examples





Diophantine approximation

Hyperfinite graph families

Family $\mathcal{G}$ of finite graphs is hyperfinite:

$$\forall \varepsilon > 0 \; \exists k = k(\varepsilon) \;\; \forall G \in \mathcal{G} \;\; \mathrm{sep}_k(G) \leq \varepsilon.$$

Hyperfinite: paths, trees, planar graphs,
every non-trivial minor-closed property

Non-hyperfinite: expanders

Hyperfinite graph families

Every graph property is testable in any family of hyperfinite graphs.

Newman – Sohler
(Benjamini-Schramm-Shapira, Elek)

Hyperfinite graph families

Every graph property is testable in any family of hyperfinite graphs.

Many graph properties are polynomial time testable for graphs with bounded tree-width.

Hyperfinite graph sequences

If $G_n \to G$ locally, then

$\{G_n\}$ is hyperfinite $\Leftrightarrow$ G is hyperfinite

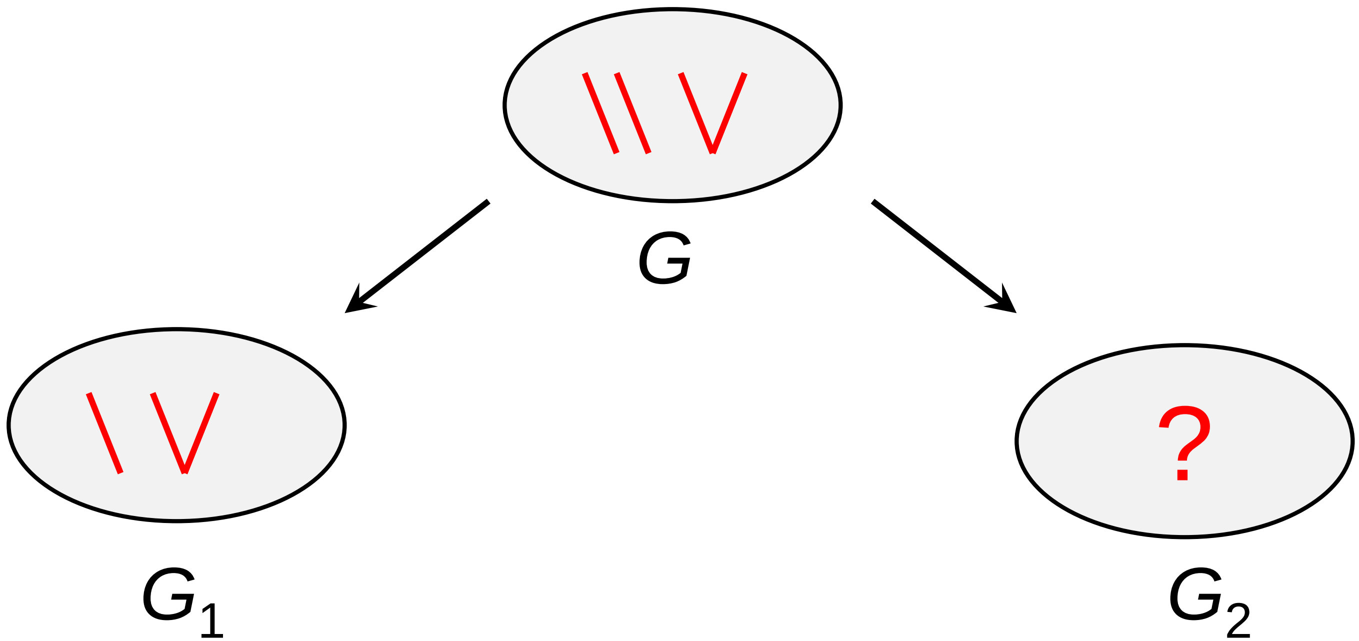
Schramm

(Benjamini-Shapira-Schramm)

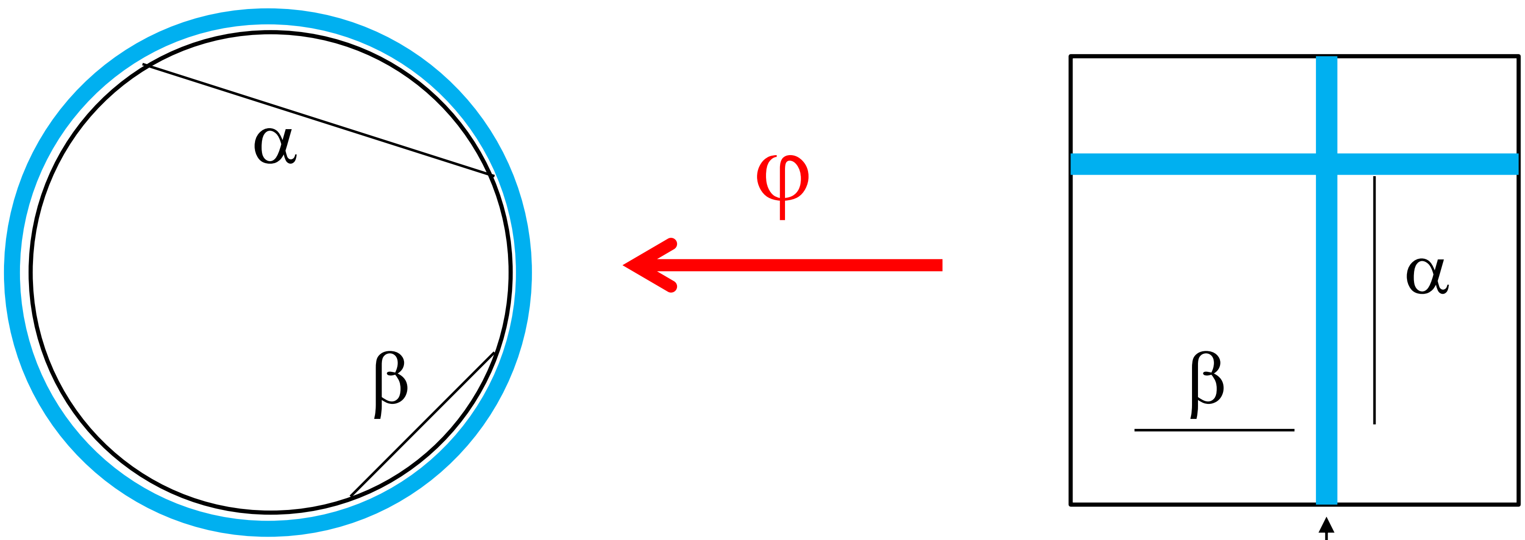
Hyperfinite graphings

If G_1 and G_2 are locally equivalent, then

G_1 is hyperfinite $\Leftrightarrow$ G_2 is hyperfinite



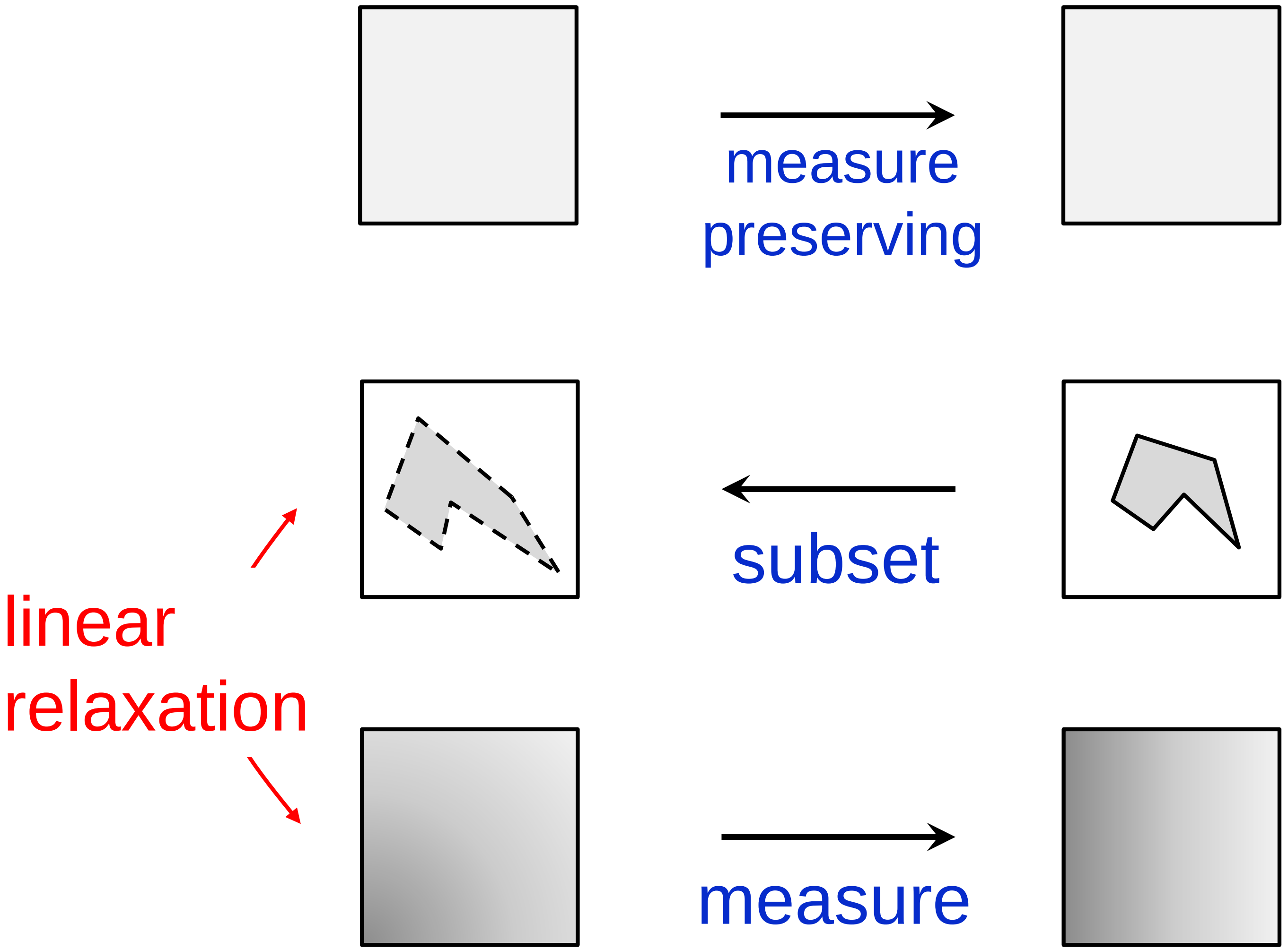
Local isomorphism forward



$$(x,y) \mapsto x+y \bmod 1$$

Diophantine approximation

Pushing forward and pulling back



Fractional graph partition problem

$$\mathcal{R}_k = \left\{ X \subseteq V(G) : G\left[X\right] \text{connected}, \left|X\right| \leq k \right\}$$

T: optimal k-edge-separator

$$\mathcal{F} = \left\{ V(H) : H \text{ comp't of } G - T \right\} \subseteq \mathcal{R}_k$$

Fractional graph partition problem

$$\sigma(Y) = \begin{cases} \frac{|Y|}{n}, \text{ if } Y \in \mathcal{F}, \\ 0, \text{ if } Y \in \mathcal{R}_k \setminus \mathcal{F} \end{cases}$$

$$\sum_{Y \in \mathcal{R}_k} \sigma(Y) = 1$$ ← probability distribution

$$\sum_{Y \ni x} \frac{\sigma(Y)}{|Y|} = \frac{1}{n}$$ ← „marginal" uniform

$$\sum_{Y \in \mathcal{R}_k} \sigma(Y) \frac{|\partial Y|}{|Y|} = 2\,\mathrm{sep}_k(G)$$ ← expected expansion

Fractional graph/graphing partition problem

Define

$$\mathrm{sep}_k^*(G) = \min_\tau \frac{1}{2} \mathrm{E}_\tau\left(\frac{|\partial Y|}{|Y|}\right)$$

Can be defined for graphings

τ probability distribution on $\mathcal{R}_k$

with uniform marginal

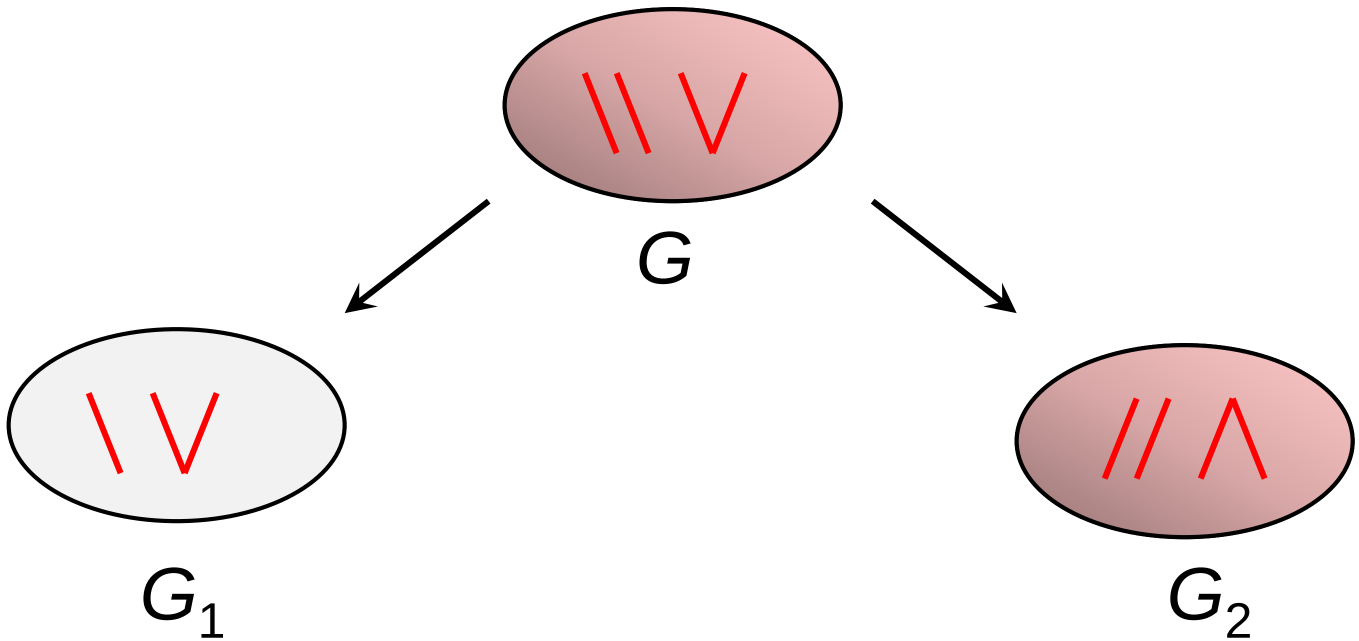
$$\mathrm{sep}_k^*(G) \le \mathrm{sep}_k(G) \le \mathrm{sep}_k^*(G) \log \frac{8D}{\mathrm{sep}_k^*(G)}$$

no dependence on k

Hyperfinite graphings

If G_1 and G_2 are locally equivalent, then

G_1 is hyperfinite $\Leftrightarrow$ G_2 is hyperfinite



Proof sketch

$$\mathrm{sep}_k(G) \le \mathrm{sep}^*_k(G) \log \frac{8D}{\mathrm{sep}^*_k(G)}$$

Algorithm: For j=1,2,..., select $Y_1, Y_2, \ldots \in \mathcal{R}_k$ so that Y_j is the minimizer of

$$\frac{|\partial Y|}{|Y \setminus (Y_1 \cup \ldots \cup Y_{j-1})|}$$

Output: $X = \partial Y_1 \cup \partial Y_2 \cup \ldots$

On a graphing:
no uncountable
sequence of steps!
Phases...

Thanks, that’s all!