To where Knuth paths may lead

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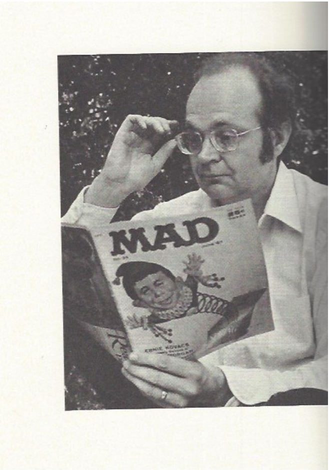
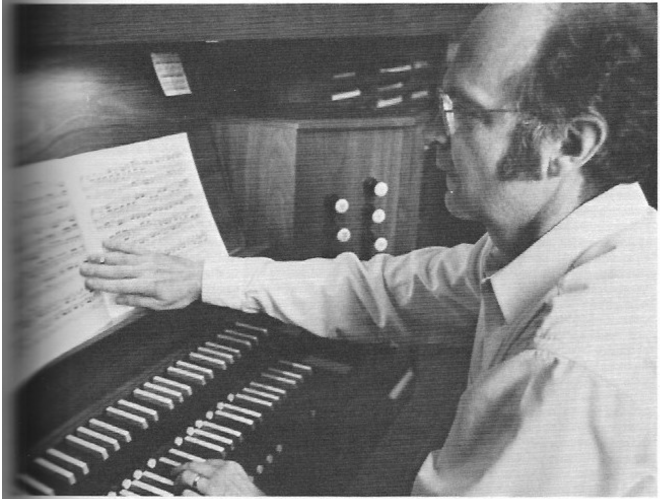
Knuth paths



- *The Art of Computer Programming*
- *Concrete Mathematics* (with R. Graham and O. Patashnik)

Knuth paths



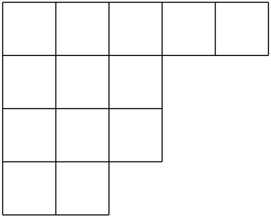
Knuth is an accomplished organist and composer. "I want to write some music for organ with computer help. If I live long enough, I would like to write a rather long work that would be based on the book of Revelation. The musical themes would correspond to the symbolism in the book of Revelation."

Young tableaux

Partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 \geq \ldots \geq \lambda_k$, $|\lambda| = \sum \lambda_i$, $\lambda \vdash n$

Represented by diagram with λ_i boxes in row i

Example: $\lambda = (5, 3, 3, 2)$, $|\lambda| = 13$



Young tableaux

1	2	3	5	13
4	7	9		
6	8	11		
10	12			

Standard Young tableau (SYT) of shape λ = filling of the n boxes of the λ diagram by entries $1, 2, \ldots, n$ so that rows and columns are strictly increasing.

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f_λ = number of standard Young tableaux of partition shape λ

Example: $f_{(3,2)} = 5$

1	2	3	1	2	4	1	2	5	1	3	4	1	3	5
4	5		3	5		3	4		2	5		2	4	

RS insertion algorithm for x=35214

Step1 :

P_i

3	

Q_i

1	

Step2 :

P_i

3	5

Q_i

1	2

Step3 :

P_i

2	5
3	

Q_i

1	2
3	

Step4 :

P_i

1	5
2	
3	

Q_i

1	2
3	
4	

Step5 :

P_i

1	4
2	5
3	

Q_i

1	2
3	5
4	

RS correspondence

$$x = 35214 \quad \leftrightarrow \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \cline{1-1} \end{array}$$

$$x \quad \leftrightarrow \quad (P(x), Q(x))$$

Theorem (Robinson-Schensted) The mapping $x \mapsto (P(x), Q(x))$ is a bijection between permutations $x \in S_n$ and pairs of standard tableaux of same shape (P, Q).

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Don [TAoCP, Vol. 3]: "RS correspondence has magical properties, one is lead to suspect witchcraft ..." For instance,

$$x^{-1} \qquad \leftrightarrow \qquad (Q(x), P(x))$$

RSK correspondence

Theorem (Knuth)

- Vast generalization: Bijections between $\mathbb{N}$-matrices and pairs of semistandard Young tableaux.

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Two important applications, among many:

- Bender-Knuth
- Kazhdan-Lusztig

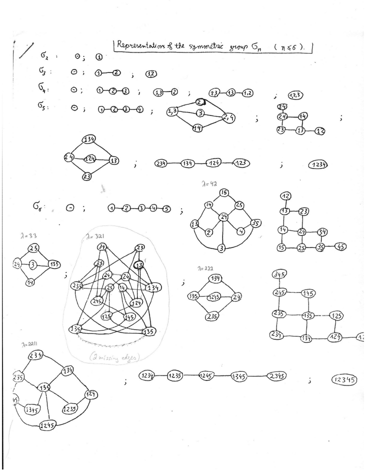
RSK correspondence

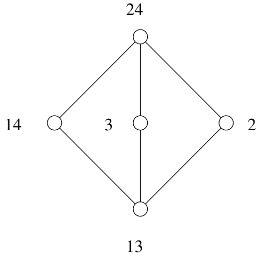
9	5	5	3	2
9	4	4		
6	3	1		
5	2			

- The Bender-Knuth bijective proof of MacMahon's amazing formula for the number of plane partitions of n:

$$\Sigma_{k\geq 0} PP(k)x^k = \frac{1}{(1-x)(1-x^2)^2(1-x^3)^3\cdots}$$

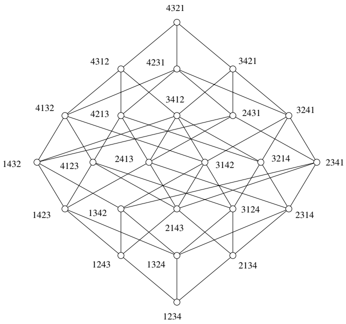
Kazhdan-Lusztig representations





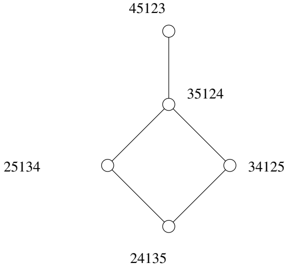
- Proof of the amazing fact that the Kazhdan-Lusztig (1979) construction of representations for the symmetric groups produces exactly the irreducible rep's, correctly labeled.

Kazhdan-Lusztig representations



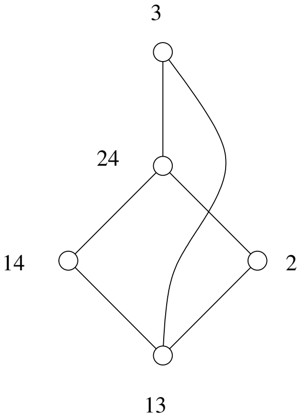
Bruhat partial order on permutations S_4, generated by inversions.

Kazhdan-Lusztig representations



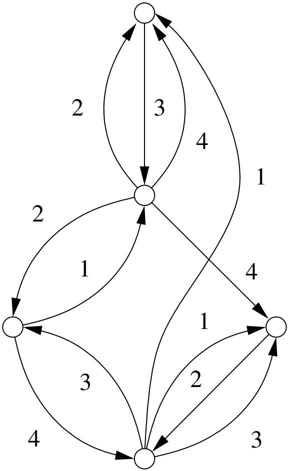
K-L cell, coming from representation of Hecke algebra (q-analog of S_n).

Kazhdan-Lusztig representations



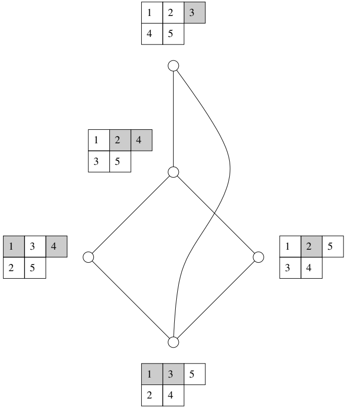
K-L polynomial, non-Bruhat edges

Kazhdan-Lusztig representations



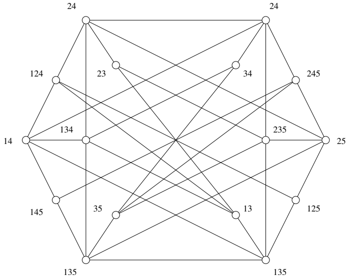
Here, bidirected double edges $\leftrightarrow$ elementary Knuth moves
$\Rightarrow$ K-L cell connnected via Knuth paths.

Kazhdan-Lusztig representations



The whole picture (almost) can via Knuth paths be translated from Hecke algebra into Knuth equivalence classes (SYT tableaux of fixed shape), etc. BUT, $\exists$ one obstacle to entirely combinatorial construction: the non-Bruhat edges depend on the K-L polynomials, and they are difficult ...

Kazhdan-Lusztig representations



The K-L graph for $\lambda = (3, 2, 1)$.
Representing integer matrices can be read off from the informaton given by graph.

The FKG inequality

Next topic: A sharper FKG inequality.

Correlation inequality, arose in statistical physics
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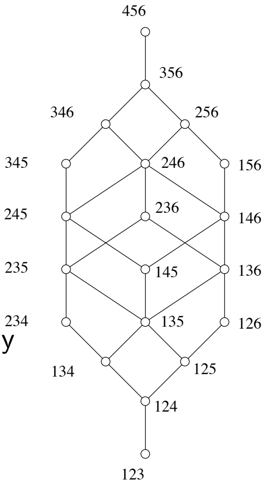
FKG inequality review

Review of concepts:

- $L = (\wedge, \vee)$ finite distributive lattice
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, and dually



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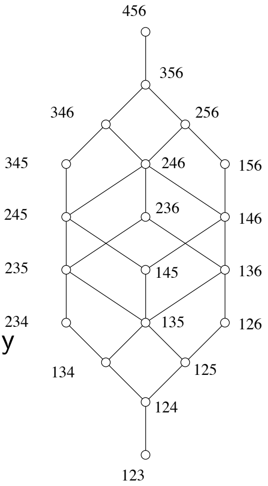

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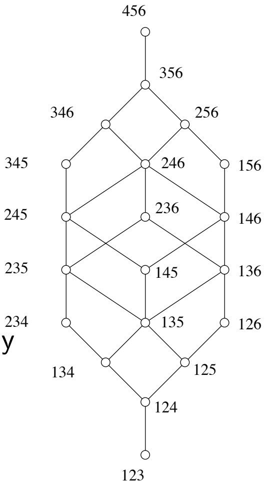

Review of concepts:

- $L = (\wedge, \vee)$ finite distributive lattice
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, and dually
- rank function $rank : L \to \mathbb{Z}^+$
- $\mu : L \to \mathbb{R}^+$ is *log-supermodular* if

$$\mu(x)\mu(y) \leq \mu(x \wedge y)\mu(x \vee y), \text{ for all } x, y \in L.$$

FKG inequality review

For functions $k, \mu : L \to \mathbb{R}$ let $E_\mu[k] \stackrel{\text{def}}{=} \sum_{x \in L} k(x)\mu(x)$.

Theorem (Fortuin, Kasteleyn and Ginibre, 1971)

Let L be a distributive lattice, $\mu : L \to \mathbb{R}^+$ log-supermodular, and g and h increasing functions $g, h : L \to \mathbb{R}^+$. Then

$$E_\mu[g] \cdot E_\mu[h] \;\leq\; E_\mu[1] \cdot E_\mu[gh].$$

Special cases of the FKG inequality

1. If $a_1 \leq \ldots \leq a_n$ and $b_1 \leq \ldots \leq b_n$, then

$$\sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i \leq n \cdot \sum_{i=1}^{n} a_i b_i$$

(Chebychev)

2. Let A and B be two families of subsets of $[n] \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$, both closed under taking subsets.

$$\frac{\|A\|}{2^n} \cdot \frac{\|B\|}{2^n} \leq \frac{\|A \cap B\|}{2^n}$$

(Kleitman)

Shows: *monotone events are positively correlated*.

A polynomial strengthening (q-analog) of FKG inequality

For functions $k, \mu : L \to \mathbb{R}$ let

$$E^q_\mu[k] \stackrel{\text{def}}{=} \sum_{x \in L} k(x)\mu(x) q^{rank(x)} \in \mathbb{R}[q].$$

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I.e., coefficient-wise inequality of real polynomials.

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Δ_1 and Δ_2 simplicial complexes on the same vertex set V.
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Theorem

$$f_{\Delta_1}(q) \cdot f_{\Delta_2}(q) \ll (1+q)^{|V|} \cdot f_{\Delta_1 \cap \Delta_2}(q)$$

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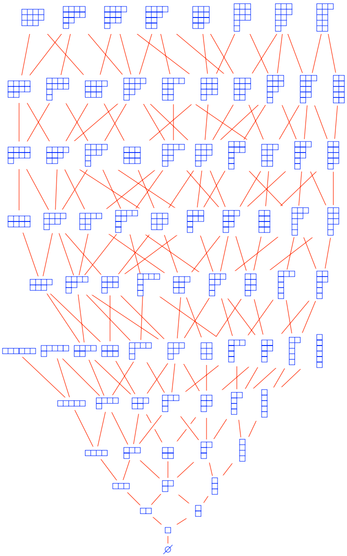
Note: $q = 1$ gives Kleitman's inequality

Appl. 2: Positive correlation wrt Plancherel measure



Young's lattice Y

Partitions ordered by inclusion of diagrams. A distributive lattice.

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For the symmetric group $G = S_n$:

- set of irreducible representations $\mathrm{Irr}(S_n) \leftrightarrow \{\text{partitions of } n\}$
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For the symmetric group $G = S_n$:

- set of irreducible representations $\mathrm{Irr}(S_n) \leftrightarrow \{\text{partitions of } n\}$
- $\deg(\lambda) = f_\lambda$, for partitions λ
- Function $\mathrm{Irr}(G) \to \mathbb{R}$ given by $\lambda \mapsto \frac{f_\lambda^2}{n!}$ known as *Plancherel measure* on $\mathrm{Irr}(G)$.
- Extend to measure on all partitions (thus on Y) by *poissonization*

$$\lambda \mapsto \pi_\theta(\lambda) \stackrel{\text{def}}{=} e^{-\theta} \ \frac{\theta^{|\lambda|} f_\lambda^2}{(|\lambda|!)^2}$$

Here $\theta > 0$ is a real parameter.

Appl. 2: Positive correlation wrt Plancherel measure

Asymtotics of these measures much studied. E.g.,

- measure concentration phenomenon known (limit shape of typical partition diagram),

Early references: Logan-Shepp, Vershik-Kerov, Borodin-Okounkov-Olshanski, ...

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- and to distribution of eigenvalues of random matrices.

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Proposition

Suppose that $0 \leq s \leq t$. *Then the function*

$$\lambda \mapsto \frac{f_\lambda^t}{(|\lambda|!)^s}$$

is log-supermodular on the Young lattice.

Proof. Use hook-length formula. $\square$

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$s = 1, t = 2$ case Plancherel measure

$s = t = 2$ case its poissonization

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Theorem

If $g, h : Y \to \mathbb{R}^+$ are increasing functions (w.r.t. containment of partition shapes), then

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Proof of Theorem. From the q-FKG inequality and log-supermodular proposition. truncating the formal power series to polynomials on lower intervals. □

Happy birthday!

At the age of three, Don Knuth was already attracted to keyboards.